M-STRUCTURE IN DUAL BANACH SPACES

BY

F. CUNNINGHAM, JR., EDWARD G. EFFROS[†], AND NINA M. ROY

ABSTRACT

A result previously known only for certain ordered Banach spaces is generalized to arbitrary real Banach spaces. Let $\mathscr{L}(U)$ be the Banach algebra of operators generated by the L-projections of a real Banach space U, and let $\mathscr{M}(U^*)$ be the bounded operators on the dual space U^* with adjoint in $\mathscr{L}(U^{**})$. Then the adjoint operation maps $\mathscr{L}(U)$ onto $\mathscr{M}(U^*)$. In particular, any M-projection of U^* is weak* continuous.

In many respects the most important nonreflexive real Banach spaces are the L-spaces (isometric to L^1 of a measure space) and the C-spaces (isometric to $C(K)$ with the uniform norm, K compact Hausdorff). These classes are mutually dual in the sense that the conjugate of a space in either class belongs to the other. More general structures in arbitrary Banach spaces, imitating the norm properties of L-spaces and C-spaces, have been studied in [1, 2, 4, 5]. Recall that a projection e of a Banach space U is called an *L-projection* if it satisfies

$$\|u\| = \|eu\| + \|u - eu\|$$

for all u in U, and an *M-projection* if instead

$$\|u\| = \max\{\|eu\|, \|u - eu\|\}.$$

The norm-closed operator algebra $\mathscr{L}(U)$ (called $\mathscr{C}(U)$ in [1, 2]) generated by the L-projections is (abstractly) a commutative real von Neumann algebra which we shall call the *L-structure* of U. The definition of M-structure needs to be more delicate because the M-projections may be too scarce. One approach [1, 2] defines an *M-ideal* in U to be a closed subspace whose annihilator in U^* is the

[†]Supported in part by the National Science Foundation.
Received July 19, 1972

range of an L-projection and the *centralizer* $\mathscr{Z}(U)$ (so-called because that is what it is when U is a C^*-algebra) to be the algebra of bounded operators on U whose adjoints belong to $\mathscr{L}(U^*)$. Another approach [5] defines an operator algebra called the *maximal M-structure* of U, which turns out to be the same as the centralizer. We denote this algebra here by $\mathscr{M}(U)$. The L-structure and the M-structure so defined are mutually dual in the sense that adjoint carries either structure of U into the other structure of U^*.

A striking asymmetry in these dualities appears when you try to go backwards against the * functor. In one direction we have the following rich, untidy situation.

i) A space U whose conjugate is an L-space need not be a C-space. The interesting profusion of spaces in this class, called Lindenstrauss spaces, have been studied, for example, in [11].

ii) An L-projection of U^* need not be the adjoint of an M-projection of U. Indeed, if K is connected, then $U = C(K)$ has no nontrivial M-projections, even though U^* is an L-space.

iii) The L-structure of U^* may contain vastly more than the adjoints of the M-structure of U. This is illustrated, of course, by the example cited in (ii) but the discrepancy goes much farther. There are Lindenstrauss spaces with trivial M-structure. (The simplex space with countable non-Hausdorff structure space described at the end of [8], and the (nonseparable) G-spaces which are not square constructed in [6] are examples.)

In the other direction, by contrast, we have theorems:

(i′) If U^* is a C-space, then U is an L-space.

(ii′) Any M-projection of U^* is the adjoint of an L-projection of U.

(iii′) The M-structure of U^* comes from the L-structure of U by the adjoint.

The first of these results is a theorem of Groethendieck [9]. The others are Theorems 1 and 2 below. They answer Problems 1 and 5 respectively posed in [2, §7]. For a large class of ordered Banach spaces, called F-spaces, Theorem 1 has been proved by Perdrizet [10], and Theorem 2 by Alfsen and Effros [2].

In what follows, U is a real Banach space, and $V = U^*$.

THEOREM 1. *If e is an M-projection of V, then $e = f^*$, where f is an L-projection of U.*

PROOF. (Adapted from [3] where it is given for the separable case.) It suffices to show that the subspaces eV and $(I - e)V$ are weak* closed (see [9, p. 556]).

Since $I - e$ is also an M-projection, we need only consider eV. Letting D be the closed unit ball of V, it suffices to prove that $eD = eV \cap D$ is weak* closed (see [7, p. 429]).

Suppose that $\{v_\gamma\}_{\gamma \in \Gamma}$ is a net in eD converging weak* to an element v in V. Since the net $v_\gamma - ev$ also lies in eV and converges to $v - ev$, we may initially assume that v lies in $(I - e)V$, and prove that $v = 0$.

Let us suppose that $v \neq 0$. Since e is an M-projection, we have for large scalars a and $\gamma \in \Gamma$,

$$\| v_\gamma + av \| = \max\{ \|v_\gamma\|, a\|v\| \} = a\|v\|.$$

On the other hand, $\{v_\gamma + av\}$ converges to $(1 + a)v$. Since the norm function is weak* lower semicontinuous (norm closed balls are weak* closed), we have

$$(1 + a)\|v\| \leqq \liminf \|v_\gamma + av\| = a\|v\|,$$

a contradiction. We conclude that $v = 0$, completing the proof.

We refer the reader to [2] for the definitions of M-codirection (denoted $|_M$) and M-domination ($\prec_M$). We say that a cone C in V is an *M-cone* if we have

(a) C is convex,

(b) $v |_M w$ for all v and w in C,

(c) If $v \prec_M w$ and $w \in C$ then $v \in C$.

An M-cone must be proper since if $v |_M - v$, then $v \prec_M v + (-v) = 0$, and $v = 0$ We note that the M-cones may be regarded as the M-structure analogues of the facial cones defined in the theory of L-structure (see [1]).

Given $v \in V$, we define

$$C_M(v) = \{w \in V : w \prec_M av \text{ for some } a \geqq 0\}.$$

Since $av |_M bv$ for $a, b \geqq 0$, we have from Lemma 4.1 of [2] that $C_M(v)$ is the smallest M-cone containing v.

LEMMA 1. *If C is an M-cone in V, then $\prec_M$ restricts to the intrinsic linear ordering on C.*

PROOF. Given $v, w \in C$ with $v \prec_M w$, we have that $w - v \prec_M w$, hence from (c), $w - v \in C$ and $v \leqq w$ with respect to C. Conversely, if $0 \leqq v \leqq w$ with respect to C, then $v, w - v \in C$ and (b) imply that $v |_M w - v$, hence $v \prec_M w$.

Given $v, w \in V$ with $v \prec_M w$, we define

$$[v, w] = \{x \in V : v \prec_M x \prec_M w\}.$$

LEMMA 2. *Given v and w in V with $v \prec_M w$, we have*

$$[v,w] = [0,w] \cap (v + [0,w]).$$

PROOF. Let $\leqq$ be the intrinsic linear ordering on $C_M(w)$. Then $[v,w] \subseteq C_M(w)$, and from Lemma 1,

$$\begin{aligned} [v,w] &= \{x \in V : v \leqq x \leqq w\} \\ &= \{x \in V : 0 \leqq x \leqq w\} \cap \{x \in V : 0 \leqq x - v \leqq w\} \\ &= [0,w] \cap (v + [0,w]). \end{aligned}$$

LEMMA 3. *Given $v \prec_M w$, the set $[v,w]$ is weak* compact.*

PROOF. From Lemma 2, it suffices to prove that $[0, w]$ is weak* compact. However, the latter is the intersection of all norm closed balls containing 0 and w. Since such balls are weak* compact, the result follows.

LEMMA 4. *Suppose that $\{v_\gamma\}_{\gamma \in \Gamma}$ is a net in V which is increasing with respect to the partial ordering $\prec_M$, and that $v_\gamma \prec_M w$ for some $w \in V$. Then v_γ converges weak* to an element v, which is the $\prec_M$ least upper bound for the set $\{v_\gamma\}$.*

PROOF. The net $\{v_\gamma\}$ lies in the weak* compact set $[0,w]$. Let v be the weak* limit of a subnet of $\{v_\gamma\}$. If $_{\gamma_0}$ is a fixed index, and w' is any $\prec_M$ upper bound for the set $\{v_\gamma\}$, we eventually have that the subnet lies in $[v_{\gamma_0}, w']$. From Lemma 3, it follows that v lies in $[v_{\gamma_0}, w']$, i.e., v is the $\prec_M$ least upper bound of the set $\{v_\gamma\}$. This property uniquely characterizes v, hence v is the only cluster point of the net $\{v_\gamma\}$. From the compactness of $[0,w]$, we conclude that v_γ converges weak* to v.

LEMMA 5. *If $\{S_\gamma\}_{\gamma \in \Gamma}$ is an increasing net in $\mathscr{M}(V)$, and $0 \leqq S_\gamma \leqq T$ for some $T \in \mathscr{M}(V)$, then there is a least upper bound S for the net $\{S_\gamma\}$.*

PROOF. The net $\{S_\gamma v\}$ is $\prec_M$ increasing for each $v \in V$, and we have $S_\gamma v \prec_M Tv$. From Lemma 4 we may define Sv to be the weak* limit of $\{S_\gamma v\}$. It is evident that S is linear and $S_\gamma \prec_M S \prec_M T$. From [2, Lemma 4.6], S lies in $\mathscr{M}(V)$, and it is clear that S is the least upper bound for the $\{S_\gamma\}$.

THEOREM 2. *The map $T \to T^*$ is an isomorphism of $\mathscr{L}(U)$ onto $\mathscr{M}(V)$.*

PROOF. From Lemma 6.11 of [2] or Theorem 5 of [5], $T \to T^*$ is an isometric isomorphism of $\mathscr{L}(U)$ into $\mathscr{M}(V)$. From Lemma 5, the spectrum of $\mathscr{M}(V)$ is extremally disconnected, hence $\mathscr{M}(V)$ is the norm closure of the algebra generated by the idempotents, i.e., the M-projections. The result thus follows from Theorem 1.

REFERENCES

1. E. Alfsen and E. Effros, *Structure in real Banach spaces, Part I*, Ann. of Math., **96** (1972), 98–128.

2. E. Alfsen and E. Effros, *Structure in real Banach spaces, Part II*, Ann. of Math., **96** (1972), 129–173.

3. F. Cunningham, Jr., *L^1-structure in Banach spaces*, Thesis, Harvard University, 1953.

4. F. Cunningham, Jr., *L-structure in L-spaces*, Trans. Amer. Math. Soc. **95** (1960), 274–299.

5. F. Cunningham, Jr., *M-structure in Banach spaces*, Proc. Cambridge Philos. Soc. **63** (1967), 613–629.

6. F. Cunningham, Jr., *Square Banach spaces*, Proc. Cambridge Philos. Soc. **66** (1969) 553–558.

7. N. Dunford and J. Schwartz, *Linear Operators*, Part 1, Interscience, New York, 1958.

8. E. Effros, *Structure in simplexes*, Acta Math. **117** (1967), 103–121.

9. A. Grothendieck, *Une caractérisation vectorielle-métrique des espaces L^1*, Canad. J. Math. **7** (1955), 552–561.

10. F. Perdrizet, *Espaces de Banach ordonnés et idéaux*, J. Math. Pures Appl. **49** (1970), 61–98.

11. J. Lindenstrauss and D. Wulbert, *On the classification of the Banach spaces whose duals are L_1 spaces*, J. Functional Analysis **4** (1969), 332–349.

BRYN MAWR COLLEGE
UNIVERSITY OF PENNSYLVANIA
ROSEMONT COLLEGE